Frame of Reference: To describe the motion of a body, we must take another body or group of bodies to be at rest and describe the motion with respect to the second body or the group of bodies. This second body on the group is called the Frame of Reference of that description. [NB. No body is actually at rest in the universe]. In the description of planetary motion, there is heliocentric view i.e. the Sun is taken to be at rest and the planets are moving in circular paths around the Sun.

Kinematical Quantities:

1. Distance travelled or path length: \( \sum \text{AD} \ [\text{ABCDE}] \)

2. Displacement: vector directing from the initial position to final position \( \vec{D} = \vec{A} \vec{B} \), \( D > S \)

3. Speed: Time rate of change of position of a particle; i.e. distance travelled divided by the time interval.

4. Velocity \( (v) \): Time rate of displacement. Average velocity equals displacement divided by time interval.

5. Acceleration \( (a) \): Time rate of change of velocity.

Speed, Velocity, Acceleration have average & instantaneous magnitudes, that may be different, such as \( v_{av} = \frac{\Delta S}{\Delta t} \), \( v_{im} = \frac{ds}{dt} \).

Kinematic Relations for Rectilinear Motions:

1. If particle is in uniform velocity, \( S = ut \), \( a = 0 \).
2. If particle is in uniform acceleration,
   \( \text{(i) } v = u + at \), \( \text{(ii) } S = ut + \frac{1}{2} at^2 \), \( \text{(iii) } v^2 = u^2 + 2as \),
   \( \text{(iv) } S_{nth} = (u + \frac{1}{2} a) (2n-1) \).
* Vertical Motion under Grav.:
  
  a) Time of ascent : \( t_a = \frac{u}{g} \).
  
  b) Maximum height achieved : \( H = \frac{u^2}{2g} \).
  
  c) Time of descent : \( t_d = \frac{u}{g} \).
  
  d) Time of flight : \( T = \frac{2u}{g} \).

* Projectile Motion:
  
  a) Maximum height : \( H = \frac{u^2 \sin^2 \theta}{g} \).
  
  b) Time of flight : \( T = \frac{2u \sin \theta}{g} \).
  
  c) Horizontal range : \( R = \frac{u^2 \sin 2\theta}{g} \).
  
  d) Maximum range : \( R_{\text{max}} = \frac{u^2}{2g} \).
  
  e) Path of a projectile : \( y = x \tan \theta - \frac{gx^2}{2u^2 \cos^2 \theta} \), \( y = x \tan \theta \left(1 - \frac{x}{R}\right) \).
  
  f) Projectile on an incline (\( \phi \)):
    \[
    x = u \cos \phi t - \frac{1}{2} gt^2 \cos^2 \phi + \frac{u^2 \sin \phi}{g} \cos \phi t^2
    \]
    
    y = u \sin \phi t - \frac{1}{2} gt^2 \sin \phi + \frac{u^2 \sin \phi \sin (\theta + \phi)}{g} \cos^2 \phi.
  
* Uniform Circular Motion:
  
  \( \dot{\theta} = \dot{\theta} \hat{n} \), \( \dot{\omega} = \omega \hat{n} \), \( \ddot{\omega} = -\omega^2 \hat{n} \).
  
  \( T = \frac{2\pi}{\omega} \), \( \omega = 2\pi n \left( n \to \text{rpm} \right) \), \( \omega = \frac{2\pi}{T} \).
  
  For non-uniform circular motion: \( \omega = \omega_0 + \alpha t \), \( \theta = \omega_0 t + \frac{1}{2} \alpha t^2 \).
  
  \( \omega^2 = \omega_0^2 + 2\alpha \theta \), \( \dot{\theta} = -\alpha \hat{m} + \alpha \dot{t} \).

* If a particle moves a distance at speed \( v_1 \) and comes back with speed \( v_2 \), then \( v_{\text{av}} = \frac{2v_1 v_2}{v_1 + v_2} \), \( v_{\text{av}} = 0 \).

* If a particle moves in two equal intervals of time at different speeds \( v_1 \) & \( v_2 \), then \( v_{\text{av}} = \frac{v_1 + v_2}{2} \).

For motion with variable acceleration, \( a = \frac{dv}{dt} \), \( a = \frac{d^2 s}{ds^2} \).

\( v = \frac{ds}{dt} \), \( s = \int v dt \), \( v = \int a dt \) should be applied.

* The locus of the focus of all parabolas described by the particles projected simultaneously from the same point with equal velocity but in different directions is a circle.

  \( H = \frac{u^2 \sin^2 \theta}{2g \cos \phi} \).
* The velocity acquired by a particle at any point in its projectile motion, is the same as acquired by a particle in falling freely from the direction to that point.

* The parabolic equation of the projectile:

\[
\left( x - \frac{u^2 \sin \theta \cos \theta}{g} \right)^2 = \left( \frac{2u^2 \cos^2 \theta}{g} \right) \left( y - \frac{u^2 \sin^2 \theta}{2g} \right).
\]

a) Latus rectum: \( \frac{2u^2 \cos^2 \theta}{g} \)

b) Coordinates of focus: \( \left( \frac{u^2 \sin \theta \cos \theta}{g}, -\frac{u^2 \sin^2 \theta}{2g} \right) \)

c) Equation of directrix: \( y = \frac{u^2}{2g} \)

* The range of \( n \)th trajectory:

\[
e^{n-1} \cdot \frac{u^2 \sin 2\theta}{g} \left[ 1 + \text{coefficient of restitution} \right].
\]

* for projectile on an inclined plane, for maximum range \( 2\theta - \phi = \frac{\pi}{2} \), \( R_{\text{max}} = \frac{u^2}{g(1+\sin \phi)} \)

* When the range of a particle on an inclined plane is maximum, the focus of the path is on the plane.

* From a point on ground at distance \( x \) from a vertical wall, a ball is thrown at an angle \( 15^\circ \); it just clears the wall and strikes the ground at a distance \( y \) on the other side. Then height of the wall is \( \frac{xy}{x+y} \).

* If a body moves along a straight line by an engine delivering constant power, then \( t \leq s^{\frac{2}{3}} \).

* If \( a, b, c \) be distances moved by a particle along the \( x, y, z \) axis respectively with uniform acceleration during \( 2^{nd}, 4^{th}, 6^{th} \) second of its motion respectively, then \( a(y-z) + b(z-x) + c(x-y) = 0 \).
Relative Velocity: \( \mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b \).

\[ \mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b \]

Crossing river problems: i) in shortest time, \( t = \frac{d}{v_b} \),

ii) displacement of boat, \( \sqrt{d^2 + \left(\frac{d}{v_b} \right)^2} \),

iii) for shortest distance, \( v_r = v_b \sin \theta \),

\[ t = \frac{d}{\sqrt{v_b^2 + v_r^2}} \]

When \( n \) number of particles are located at the vertices of a regular polygon of \( n \) sides having side length \( a \), if they start moving heading to each other with speed \( v \), they must collide at the center of the polygon after the time,

\[ t = \frac{a}{v \left(1 - \cos \frac{2\pi}{n}\right)} \]

Velocity of approach: If two particles \( A \) and \( B \) separated by a distance \( a \) at a certain instant of time move with velocities \( v_1 \) and \( v_2 \) at angles \( \theta_1 \) and \( \theta_2 \) with the direction \( \vec{AB} \), the velocity by which the particle \( A \) approaches \( B = v_1 \cos \theta_1 - v_2 \cos \theta_2 \).

Angular velocity of \( B \) with respect to \( A = \left(\frac{v_2 \sin \theta_2 - v_1 \sin \theta_1}{a}\right) \).

When particle covers one-third distance at speed \( v_1 \), next one-third at \( v_2 \), last one-third at speed \( v_3 \), then \( v_{av} = \frac{3v_1 v_2 v_3}{v_1 v_2 + v_2 v_3 + v_3 v_1} \).

\[ v_{av} = \frac{d_1 + d_2 + d_3 \ldots}{n} \]

\[ a = \frac{a_1 a_2}{a_1 + a_2} \]

A particle moving with uniform acceleration from \( A \) to \( B \) along a straight line has velocities \( v_1 \) and \( v_2 \) at \( A \) & \( B \) respectively. If \( C \) is the mid-point between \( A \) & \( B \), then velocity of the particle at \( C \) is equal to

\[ v = \sqrt{\frac{v_1^2 + v_2^2}{2}} \]
A particle is dropped vertically downwards from rest. The time taken by it to fall through successive distances of 1 m each will then be in the ratio of the difference in the square roots of the integers, i.e., \( \sqrt{11} : \sqrt{12} - \sqrt{11} : \sqrt{13} - \sqrt{12} : \sqrt{14} - \sqrt{13}, \ldots \)

If \( a = f(t) \) then \( v = u + \int f(t) \, dt \) \( s = ut + \int (f(t) \, dt) \)

If \( a = f(x) \) then \( v^2 = u^2 + 2 \int f(x) \, dx \)

If \( a = f(v) \) then \( t = \int \frac{v}{f(v)} \, dv \) and \( s = x_0 + \int \frac{v \, dv}{f(v)} \)

Equation of instantaneous velocity of projectile:

\[
\nu_t = \sqrt{v^2 + g^2 t^2 - 2gv \sin \theta}, \quad \tan \phi = \frac{v_t}{u \cos \theta}
\]

Angular momentum of projectile at highest point of trajectory about the point of projection is

\[
L = m \cdot u \cos \theta \cdot \frac{u^2 \sin^2 \theta}{g}
\]

If \( B \) and \( C \) are at the same level on the trajectory & the time difference between these two points is \( t_2 - t_1 \), similarly \( A \) and \( D \) are also at the same level and the time difference between these two positions is \( t_2 - t_1 \), then \( t_2^2 - t_1^2 = \frac{8h}{g} \).

If \( R = 4H \cot \theta \), then \( \theta = 76^\circ \).

\[
y = \frac{R^2 t^2}{2}, \quad t_1, t_2 = \frac{R \sin \theta}{g} \left\{ 1 \pm \sqrt{1 - \left(\frac{2gy}{R \sin \theta} \right)^2} \right\}
\]

Motion of a projectile observed from another projectile:

\[
a_f(x_1 - x_2) = (u_1 \cos \theta - u_2 \cos \theta_2) \, t +
\]

\[
y = (y_1 - y_2) = (u_1 \sin \theta - u_2 \sin \theta_2) \, t
\]

For projectiles, KE = \( \left( \frac{1}{2} m u^2 \right) \cos^2 \theta \)

PE = \( \left( \frac{1}{2} m u^2 \right) \sin^2 \theta \).
Horizontal Projectile:
\[ \alpha = ut \]
\[ y = \frac{1}{2} a t^2 = \frac{g x^2}{2u^2} \]
\[ n = u t \left( 1 - \frac{g t^2}{2u^2} \right)^{1/2} \]

**U-Parabola equation:**
\[ \sqrt{u^2 + 2g y} \]
\[ \theta = \tan^{-1} \left( \frac{g}{2u} \right) \] = angle of \( u \) from horizontal.
\[ T = \sqrt{\frac{2h}{g}} \] (height \( h \))
\[ R = u \sqrt{\frac{2h}{g}} \]

For projectile on an inclination, maximum range occurs when \( \theta = \frac{\pi}{4} - \frac{\phi}{2} \)

Max. range along the inclined plane when the projectile is thrown upwards is given by \( \frac{u^2}{g(1 + \sin \phi)} \)
when downwards \( \frac{u^2}{g(1 - \sin \phi)} \)

For circular motion

\[ |\Delta \vec{v}| = 2\pi rm \frac{v}{2} \]

Skidding of vehicle on a level road, \( v_{safe} = \sqrt{\frac{m g}{g}} \)

Bending of cyclist, \( \tan \theta = \frac{v^2}{g} \)

Banking of road, \( \tan \theta = \frac{v^2}{g} \quad \text{or} \quad \frac{u^2}{g} = \frac{1}{1 + \mu \tan \theta} \)

Friction present, \( \frac{x^2}{g} = \frac{\mu + \tan \theta}{1 - \mu \tan \theta} \)

\( v_{safe} = \sqrt{\frac{g v}{h}} \)

Overturning of car on flat road, \( v_{safe} = \sqrt{\frac{g v}{h}} \)

Charged particle in magnetic field,
\[ r = \frac{mv}{q B} \]

Motion in vertical circle:
\[ v = \sqrt{u^2 - 2gh} = \sqrt{u^2 - 2gh(1 - \cos \theta)} \]
\[ \text{Tension} = \frac{m}{g} \left( u^2 - g \left( 2 - 3 \cos \theta \right) \right) \]
Cortical velocity, \( u = \sqrt{sg} \)
Kinematics.

* Motion of block on frictionless hemispherc.
  \[ v = \sqrt{2gh_0 h} \quad \{ h = \frac{a}{2} \, n \} \] → losing contact from hemispherc.

* Conical Pendulum: Tension = \( \frac{mg^2}{\sqrt{I^2 + \omega^2}} \). \( \tan \omega = \frac{v}{c} \)
  \[ T_p = 2\pi \sqrt{\frac{m}{g \tan \omega}} \]
  \( v = \sqrt{\frac{mg}{\tan \omega}} \)
  \( \omega = \sqrt{\frac{g}{\tan \omega}} \).

* Circular Motion:

1) If a tube filled with an incompressible fluid of mass \( m \) and closed at both ends is rotated with \( \omega \), force exerted by liquid at the other end = \( \frac{1}{2} \, ml \omega^2 \).

2) When a particle describes a horizontal circle on the smooth inner surface of a conical funnel such that the height of the plane of circle above the vertex is \( h \), the speed of particle \( v = \sqrt{gh} \).

3) \( \beta \), radius of curvature =

\[ \frac{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}}{\left( \frac{d^2 y}{dx^2} \right)} \]

\[ \beta = \frac{mv^3}{| \mathbf{F} \times \mathbf{v} |} \]

v) for projectile motion, \( \beta = \frac{u^2 \cos \theta}{g} \)

v) for vertical circular motion,

upper most point, \( v_c = \sqrt{\frac{g}{\cos \theta}} \)

lower most point, \( v_c = \sqrt{\frac{g}{\sin \theta}} \).