Simple Harmonic Motion

* Periodic Motion: motion of a body repeating itself over an interval of time. (e.g. planetary motion around sun, pendulum motion etc.)

* Oscillation/Vibration: Those periodic motions, in which during half of the time period the body goes in one direction and during the next half it goes in the opposite direction along the same path, are called oscillations. All oscillations are periodic, but all periodic motions are not oscillations.

* Simple Oscillatory Motion: The vibration that takes place along a straight line.

* Simple Harmonic Motion: A periodic oscillatory motion in which acceleration of the oscillating body is proportional to the distance from the mean position and acceleration is always directed towards the mid-point.

\[ a = -x, \quad f_\alpha - x \Rightarrow F = -kx. \]

\[ K \rightarrow \text{force constant, } \quad \text{spring constant}. \]

* A body in a stable equilibrium is capable of vibrating.
\( x = -a \quad \quad x = 0 \quad \quad x = +a \)

\[
\begin{align*}
KE &= 0 & \max & & 0 \\
PE &= \max & & 0 & \max \\
F &= \max & & 0 & \max \\
Acc. &= \max & & 0 & \max \\
v &= 0 & \max & 0 & 0 
\end{align*}
\]

* \( F = -kx \Rightarrow m \frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2x \)

\[
\frac{k}{mv} = \omega^2 
\]

- basic eqn of STHM

\[
\left[ \frac{d^2}{d\theta^2} (\sin \theta) = -m^2 \sin \theta, \quad \frac{d^2}{d\theta^2} (\cos \theta) = -m^2 \cos \theta \right]
\]

by inspection, \( x = a \sin \omega t \) or \( x = a \cos \omega t \).

more generally, \( x = a \sin (\omega t + \theta) \)

\[ x = a \cos (\omega t + \theta) \]

- distance from amplitude phase angle/mean position of phase of vibration at time + vibration. \( \delta \rightarrow \) initial phase/epoch of the vibration.

\[
T = \frac{2\pi}{\omega} 
\]

\[
\begin{align*}
x &= a \sin (\omega t + \delta) = a \sin [\omega (t + \tau) + \delta] \\
In \ a \ time \ interval \ t, \ phase \ changes \ by \ \frac{2\pi}{T}x \\
n, \ frequency = \frac{\omega}{2\pi} \Rightarrow [\omega = 2\pi n] \\
\omega \rightarrow \text{angular frequency, } \text{pulsation.} \\
\end{align*}
\]

- Most general eqn of STHM:

\[
x = a \sin (\omega t + \delta_1) + b \sin (\omega t + \delta_2). 
\]

* Amplitude of vibration depends on the initial displacement or thrust given to the system to start the vibration.
* Value of \( s \), initial phase depends on the displacement of the particle at the instant we start our observation.

* The projection of a uniform circular motion on any diameter of the circle is simple harmonic.

* Different physical quantities of the simple harmonic motion executing particle:
  
  \[ \begin{align*}
  1. v &= a \omega \cos(\omega t + \phi) = \omega \sqrt{a^2 - x^2} \\
  2. p &= m a \omega \sin(\omega t + \phi) = m \omega \sqrt{a^2 - x^2} \\
  3. a &= -\omega^2 a \sin(\omega t + \phi) = -\omega^2 x \\
  4. F &= -m \omega^2 a \cos(\omega t + \phi) = -m \omega^2 x \\
  5. KE &= \frac{1}{2} m \omega^2 (a^2 - x^2) \\
         &= \frac{1}{2} m \omega^2 a^2 \cos^2(\omega t + \phi) \\
         &= \frac{1}{2} m \omega^2 a^2 \sin^2(\omega t + \phi) \\
  6. PE &= \frac{1}{2} m \omega^2 x^2 \\
  7. E &= \frac{1}{2} m \omega^2 a^2 = \text{constant, thus total energy is always constant for the simple harmonic motion.}
  \end{align*} \]

* Variation of distance - velocity:
  
  \[ \frac{x^2}{a^2} + \frac{\dot{x}^2}{\omega^2 a^2} = 1. \]

** Plot of \( \omega \) against \( x \). From this, the change in the particle's velocity from the maximum extreme position changes from \( \omega a \) to 0.
Variation of momentum with distance:
\[ \frac{x^2}{a^2} + \frac{p^2}{m^2 \omega^2 a^2} = 1. \]

Variation of acceleration with distance:
\[ f = -\omega^2 x, \quad \tan \theta = -\omega^2 \]

Variation of force with distance:
\[ F = -m\omega^2 x, \quad \tan \theta = -\omega^2 \]

Variation of KE & PE with distance:
\[ KE = \frac{1}{2} m (\omega^2)(a^2 - x^2), \quad PE = \frac{1}{2} m\omega^2 x^2. \]

The curves terminate at \( \pm a \).

Examples of different vibrating systems:

1. Simple Pendulum:
\[ f_{res} = -mg \sin \theta, \quad -mg \frac{x}{L}. \]
\[ T = 2\pi \sqrt{\frac{L}{g}} \quad [\theta \leq 40^\circ]. \]

Generally, for larger \( \theta \\
\[ T = 2\pi \sqrt{\frac{L}{g}} \left( 1 + \frac{L}{2} \frac{\sin^2 \alpha}{2} + \frac{L^2}{4} \frac{\sin^4 \alpha + \ldots} \right) \]
\( \alpha \rightarrow \) angular amplitude (maximum)

If \( \alpha = 15^\circ \), time period calculated from the above approximate formula differs from the actual period by less than 0.9%.
2. Vertical oscillation of a liquid column in a U-tube:

\[ F_{res} = -AP \right \downarrow \left \uparrow g \quad \text{\textit{I}} = 2\pi \sqrt{\frac{\ell}{2g}} \]

(\ell \rightarrow \text{length of liquid column})

3. Vertical oscillation of a body floating in a liquid: \[ F_{res} = -APg \]

\[ T = 2\pi \sqrt{\frac{\ell}{g}} \quad \text{[\ell \rightarrow \text{length inside the liquid}]} \]

4. Oscillation of a body dropped into a tunnel across the Earth:

\[ F_{res} = -g \frac{4}{3} \pi \rho \text{min} \]

\[ T = 2\pi \sqrt{\frac{R}{g}} \quad \text{[R \rightarrow \text{radius of the Earth}]} \]

5. Oscillation of a piston of a kind of gas:

i) if isothermal process:

\[ F_{res} = -\frac{pA^2}{V} \right \downarrow \left \uparrow x \quad \text{\textit{I}} = 2\pi \sqrt{\frac{HV}{pA^2}} \]

ii) if adiabatic:

\[ F_{res} = -\frac{\sigma pA^2}{V} \right \downarrow \left \uparrow x \quad \text{\textit{I}} = 2\pi \sqrt{\frac{HV}{\sigma pA^2}} \]

6. Oscillation of a torsional pendulum:

\[ \tau_{res} = I \frac{d^2 \theta}{dt^2} = -c \theta \quad \text{[c \rightarrow torsional constant of the wire]} \]

\[ T = 2\pi \sqrt{\frac{I}{c}} \quad \text{[I \rightarrow MOI of disc]} \]

7. Physical Pendulum:

\[ \tau_{res} = -mgL \theta \]

\[ T = 2\pi \sqrt{\frac{L}{mgL}} \]
Oscillation of Spring-Mass system:

\[ F_{res} = -kx \quad T = 2\pi \sqrt{\frac{m}{k}} \]

i) \[ \frac{k_1}{m_1} + \frac{k_2}{m_2} \quad T = 2\pi \sqrt{\frac{m k_1 k_2}{k_1 + k_2}} \]

ii) \[ \frac{m}{m_1} + \frac{m}{m_2} \quad T = 2\pi \sqrt{\frac{m}{k_1 + k_2}} \]

Cutting a spring changes \( k \):

\[ \frac{x_1}{x_l} = \frac{a}{l} \quad F = k_1 x_1 = k_2 x_1 = k\frac{x_1}{x_l} a \quad k_1 = k_2 \frac{a}{x_l} \]

**Superposition of SHM's**: When two or more simple harmonic motions superpose, the resultant displacement at any instant is the sum of individual displacements at the instant. (Superposition Principle).

a) **Superposition of two SHM's having the same frequency \& phase but different amplitudes along the same line**:

\[ x_1 = a_1 \sin(\omega t + \theta) \quad x_2 = a_2 \sin(\omega t + \phi) \]

\[ x = x_1 + x_2 = (a_1 + a_2) \sin(\omega t + \theta) \]

b) **Superposition of 2 SHM's having same frequency but different phases and amplitudes**:\n
\[ x_1 = a_1 \sin \omega t \quad x_2 = a_2 \sin (\omega t + \phi) \]

\[ x = x_1 + x_2 = (a_1 + a_2 \cos \phi) \sin \omega t + a_2 \sin \phi \cos \omega t \]

\[ a_1 + a_2 \cos \phi = A \cos \delta \quad a_2 \sin \phi = A \sin \delta \]

\[ x = A \sin(\omega t + \phi) \quad A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos \phi)^{1/2} \]

\[ \tan \delta = \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi} \]
e) Superposition of 2 sins of slightly different frequencies but of the same amplitude & phase along the same line:

\[ \alpha_1 = a \sin \omega_1 t \quad \alpha_2 = a \sin \omega_2 t. \]

\[ \alpha = 2a \cos \left( \frac{\omega_1 - \omega_2}{2} \right) t \cdot \sin \left( \frac{\omega_1 + \omega_2}{2} \right) t. \]

\[ = A \sin \left( \frac{\omega_1 + \omega_2}{2} \right) t. \quad \quad A = 2a \cos \left( \frac{\omega_1 - \omega_2}{2} \right) t. \]

\[ = 2a \cos \frac{\pi}{n} t. \]

\[ A_{\text{max}} = 2a, \quad \text{rctn} = \frac{\pi}{n}, \quad t = \frac{b}{n}. \]

\[ A_{\text{min}} = 0, \quad \text{rctn} = (2p+1) \frac{\pi}{2} \quad t = \frac{2p+1}{2n}. \]

d. Superposition of 2 sins of the same frequency but different amplitudes along two lines at right angle:

i) \( \delta = \frac{\pi}{2} \)

\[ \alpha = a \sin \omega t, \quad y = b \sin (\omega t + \frac{\pi}{2}). \]

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad x^2 + y^2 = 1 \quad (a = b) \]

Resultant vibration are not simple harmonic, but periodic.

ii) \( \delta = 0 \)

\[ x = a \sin \omega t, \quad y = b \cos \omega t. \]

\[ y = \frac{b}{a} x. \quad \text{Distance from origin,} \quad r = \sqrt{a^2 + b^2} \sin \omega t. \]

Resultant motion is simple harmonic having same frequency but amplitude is \( \sqrt{a^2 + b^2} \).
\[ g = \frac{\pi}{2} \]
\[ x = a \sin \omega t \]
\[ y = b \sin (\omega t + \phi) = -b \cos \omega t \]
\[ y = -\frac{b}{a} x \]
\[ \omega = \sqrt{a^2 + b^2} \sin \omega t \]

Resultant motion is simple harmonic.

Superposition of SHM & Periodic Motion:

All complicated periodic motions are analysed as a series of 'shm' motions.

Damped SHM: When no friction against the motion, it's called free vibration.

The eqn of motion for free vibration is:
\[ m \frac{d^2 x}{dt^2} = -kx \]

For damped vibrations, damping force, \( F_d = Dv \)
\((D \rightarrow \text{constant}, v \rightarrow \text{velocity})\).

\[ F_d = \frac{D}{m} \frac{dx}{dt} \quad (b \rightarrow \text{damping factor}) \]

If \( F_d \) is small, two effects on vibration:

1. Reduction of frequency; new frequency, \( \omega_d = \sqrt{\omega^2 - b^2} \)
2. Decrease of amplitude; \( a_t = a_0 e^{-bt} \) \((a_0 \rightarrow \text{cons.})\)

If \( F_d \) is large, then a) critical damping and then with larger \( F_d \); b) overdamping happens. Body returns to the equilibrium position without oscillation.

- Total energy for damped oscillator, \( E = \frac{1}{2} ma_0^2 e^{-2bt} \)
- \( \omega_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \)
Equation of damped SHM: \( m\frac{d^2x}{dt^2} = -kx - D \frac{dx}{dt} \)

\( \Rightarrow m\frac{d^2x}{dt^2} = -\omega^2 x - 2b \frac{dx}{dt} \)

Solution of this equation, \( x = a_0 e^{-bt} \sin (\sqrt{\omega^2 - b^2} \cdot t + \phi) \)

**Forced Vibrations:** Simple harmonic force, \( F = F_0 \sin \omega t \)

\( m' = \frac{1}{T'} = \frac{\omega'}{2\pi} \). If applied on a body, vibration will occur with two frequencies, \( \omega \) & \( \omega' \). Eventually, due to loss of energy for damping, body will oscillate only with \( \omega' \). This is called forced vibration. Equation:

\[
\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \omega^2 x = \frac{F_0}{m} \sin \omega t
\]

Solution:

\[
\begin{align*}
\alpha_1 &= a_0 e^{-bt} \sin (\sqrt{\omega^2 - b^2} \cdot t + \phi) \quad \text{[Initial transient situation]} \\
\alpha_2 &= \frac{F_0/m}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2 \omega'^2 \omega^2}} \sin (\omega' t + \phi)
\end{align*}
\]

**Resonance:** If the frequency of the external force is equal to the natural frequency, then body starts vibrating with large amplitude. This is called resonance. Maximum amplitude:

\[
\frac{F_0/m}{2bw'}
\]
* For pendulum, if length is comparable to the radius of the Earth, $T = 2\pi \sqrt{\frac{1}{g} \left( \frac{1}{L} + \frac{1}{R} \right)}$

- a) $L \ll R$, $\frac{1}{L} \gg \frac{1}{R}$, $T = 2\pi \sqrt{\frac{1}{g}}$
- b) $L \to \infty$, $T = 2\pi \sqrt{\frac{R}{g}} \approx 84.6$ mins.
- c) $L = R$, $T = 2\pi \sqrt{\frac{R}{2g}} \approx 1$ hour
- d) $L = 1$ m, $T = 2$ seconds, a second pendulum

* Reduced Mass System:

\[
\begin{align*}
\ddot{x}_1 &= \frac{K}{m_1} \dot{x}_1 \\
\ddot{x}_2 &= \frac{K}{m_2} \dot{x}_2 \\
\end{align*}
\]

\[
\omega^2 = -\left( \frac{K}{m_1} + \frac{K}{m_2} \right) \\
\omega_{osc} = \sqrt{\frac{K}{\mu}} \\
\mu = \frac{m_1 m_2}{m_1 + m_2}
\]

\[
T = 2\pi \sqrt{\frac{\mu}{K}}
\]

* SHM of a particle attached to a massive spring: If the spring has length $l$ and mass $m$ and attached mass $M$, then $T = 2\pi \sqrt{\frac{M + m^2}{k}}$

* Oscillation on a curved surface:

\[
T = 2\pi \sqrt{\frac{R - m}{g}}
\]

* Rolling without slipping:

Time period of oscillation $= 2\pi \sqrt{\frac{3m}{2k}}$